数理解析研究所講究録
第 1658 巻 2009 年 124-135

On the Woronowicz's twisted product construction of quantum groups, with comments on related cubic Hecke algebra. *

JANUSZ WYSOCZAŃSKI †
Institute of Mathematics, Wrocław University
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

Abstract

We study the construction of compact quantum groups, based on the method invented by Woronowicz [SLW3], which uses a *twisted determinant*. As an example Woronowicz considered the function $S_N \ni \sigma \mapsto \mathrm{inv}(\sigma)$, where $\mathrm{inv}(\sigma)$ is the number of inversions in the permutation σ. Our twisted determinant is related to the function $S_N \ni \sigma \mapsto c(\sigma)$, where $c(\sigma)$ is the number of cycles in a permutation σ. For $N = 3$ it gave the quantum group $U_q(2)$. Here we show how the construction works if $N = 4$. We also describe the cubic Hecke algebra, associated with the quantum group $U_q(2)$.

1 Introduction

In [SLW3] Woronowicz provided a general method for constructing compact matrix quantum groups. The method depends on finding an N^N-element array $E = (E_{i_1,\ldots,i_N})_{i_1,\ldots,i_N=1}^N$ of complex numbers, called *twisted determinant*, which is (left and right) non-degenerate. Theorem 1.4 of [SLW3] says that if a C^*-algebra $\mathcal{A}$, is generated by N^2 elements u_{jk} which satisfy the unitarity condition:

$$\sum_{r=1}^{N} u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^{N} u_{jr} u_{rk}^*$$

and the following twisted determinant condition:

$$\sum_{k_1,\ldots,k_N=1}^{N} u_{j_1 k_1} \ldots u_{j_N k_N} E_{k_1,\ldots,k_N} = E_{j_1,\ldots,j_N} I$$

*Research partially supported by the European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389, the Polish Ministry of Science's research grants 1P03A01330 and N N201 270735

†e-mail: jwys@math.uni.wroc.pl

and if the array E is non-degenerate, then $(\mathcal{A}, u)$ is a compact matrix quantum group, where $u = (u_{jk})_{j,k=1}^N$. Woronowicz described the following example. For $\mu \in (0,1]$, he defined

$$E_{i_1,\dots,i_N} = (-\mu)^{\mathrm{inv}(\sigma)} \quad \text{if} \quad \sigma = \begin{pmatrix} 1 & 2 & \dots & N \\ i_1 & i_2 & \dots & i_N \end{pmatrix} \in S_N$$

is a permutation (S_N denotes the set of permutations of $\{1, 2, \dots, N\}$) and $E_{i_1,\dots,i_N} = 0$ otherwise. Here, for a permutation $\sigma \in S_N$, $\mathrm{inv}(\sigma)$ is the number of inversions of σ, which is the number of pairs (j,k) such that $j < k$ and $i_j = \sigma(j) > \sigma(k) = i_k$. Then as $(\mathcal{A}, u)$ one gets the quantum group $S_\mu U(N)$, called the *twisted* $SU(N)$ group.

In [W3] we considered another array E for $N = 3$, related to the number of cycles in a permutation. It was defined for a parameter $0 < q < 1$ as follows:

$$E(i,j,k) = \begin{cases} (-q)^{3-c(i,j,k)} & \text{if} \quad \{i,j,k\} = \{1,2,3\} \\ 0 & \text{otherwise} \end{cases}.$$

Here $c(i,j,k)$ is the number of cycles of the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$$

(which makes sense if and only if $\{i,j,k\} = \{1,2,3\}$). Then, following the Woronowicz's scheme, we obtained a quantum group, which turned out to be $U_q(2)$, the quantum deformation of the unitary 2×2 group. Moreover, the construction provided a description of it as a *twisted product* of it's quantum subgroups

$$U_q(2) = SU_q(2) \ltimes_\sigma U(1)$$

with the $*$-isomorphism $\sigma : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_2 \otimes \mathcal{A}_1$ given by

$$\sigma(1 \otimes v) = v \otimes 1, \quad \sigma(a \otimes v^k) = v^k \otimes a, \quad \sigma(c \otimes v^k) = v^{k-1} \otimes c.$$

The natural continuation of the construction given in [W3], was investigating the cases $N \geq 4$. However, as shall see below, after some tiresome computations it turned out that for $N = 4$ (and thus also for all $N \geq 4$) the quantum group we obtain (via the Woronowicz's theorem) is classical abelian.

Regarding the quantum group $U_q(2)$, we shall present also a construction of a cubic Hecke algebra. In [SLW3] Woronowicz showed that there are Hecke algebras associated with the quantum groups $SU_q(N)$, for every $N \in \mathbb{N}$, $N \geq 2$. The Hecke algebra $H_{q,n}$ described the intertwining operators for the n^{th} tensor power of the fundamental representation of the group. In this note we shall show similar construction for $U_q(2)$. The construction depends on defining an operator $\alpha : \mathbb{C}^3 \otimes \mathbb{C}^3 \mapsto \mathbb{C}^3 \otimes \mathbb{C}^3$, which satisfies the Yang-Baxter equation (3.1). The operator is not self-adjoint (contrary to the $SU_q(N)$ cases), although its square is so ($\alpha^2 = (\alpha^*)^2$). Nevertheless, it satisfies a generalization of the Hecke equation, namely $(\alpha^2 - I)(\alpha + q^2 I) = 0$ (see (4.1)). Therefore the operators $h_j := I_j \otimes \alpha \otimes I_{n-j-2}$, defined for $j = 1, \dots, n-2$, generate a *cubic Hecke algebra* (Theorem 4.3).

The paper is organized as follows. In Section 2 we give the computation showing the generalization of our $U_q(2)$ construction, for $N = 4$. Then, in Section 3, we give the construction of the operator α, and show that it satisfies thre Yang-Baxter equation. The last Section 4, contains the construction of the cubic Hecke algebra, associated with $U_q(2)$. In particular, we show there that α satisfies the cubic equation.

2 The construction associated with E

Let $N_4 = \{(i,j,k,l) : \{i,j,k,l\} \subset \{1,2,3,4\}\}$, let $E : N_4 \mapsto \mathbb{C}$ be zero outside $S_4 \subset N_4$, where the inclusion is given by $(i,j,k,l) \mapsto \begin{pmatrix} 1 & 2 & 3 & 4 \\ i & j & k & l \end{pmatrix}$ if $\{i,j,k,l\} = \{1,2,3,4\}$, and, for $0 < q < 1$, let the (non-zero) values of E (with the notation $E((i,j,k,l)) = E_{ijkl}$) be given by the function

$$S_4 \ni \sigma \mapsto (-q)^{4-c(\sigma)}.$$

Explicitely, it can be written in the following way:

$$\begin{array}{llllll} E_{1234} = 1 & E_{1243} = -q & E_{1324} = -q & E_{1342} = q^2 & E_{1423} = q^2 & E_{1432} = -q \\ E_{2134} = -q & E_{2143} = q^2 & E_{2314} = q^2 & E_{2341} = -q^3 & E_{2413} = -q^3 & E_{2431} = q^2 \\ E_{3124} = q^2 & E_{3142} = -q^3 & E_{3214} = -q & E_{3241} = q^2 & E_{3412} = q^2 & E_{3421} = -q^3 \\ E_{4123} = -q^3 & E_{4132} = q^2 & E_{4213} = q^2 & E_{4231} = -q & E_{4312} = -q^3 & E_{4321} = q^2 \end{array} \tag{2.1}$$

The function $S_4 \ni \sigma \mapsto 4 - c(\sigma) = t(\sigma)$ counts the *number of transpositions* in σ. It follows from [SLW3], Theorem 4.1, that this way we obtain a compact quantum group $(\mathcal{A}, \mathbf{u})$, where $\mathcal{A}$ is the C^*-algebra generated by 16 matrix elements $\{u_{jk} : 1 \le j,k \le 4\}$ of $\mathbf{u}$, which satisfy the unitarity condition:

$$\sum_{r=1}^{4} u_{jr}^* u_{rk} = \delta_{jk} I = \sum_{r=1}^{4} u_{jr} u_{rk}^* \tag{2.2}$$

and the twisted determinant condition:

$$\sum_{i,j,k,l=1}^{4} u_{\alpha i} u_{\beta j} u_{\gamma k} u_{\delta l} E_{ijkl} = E_{\alpha\beta\gamma\delta} I \tag{2.3}$$

for each $\{\alpha,\beta,\gamma,\delta\} \subset \{1,2,3,4\}$. The matrix $\mathbf{u} = (u_{jk})_{j,k=1}^4$ is the fundamental unitary co-representation of the quantum group. In our case the co-representation $\mathbf{u} = (u_{kl})_{k,l=1}^4$ is reducible by the following reason. The operator $P = (E^* \otimes I)(I \otimes E)$, which acts on $\mathbb{C}^4$, intertwines the fundamental representation with itself: $(P \otimes I)\mathbf{u} = \mathbf{u}(P \otimes I)$. Moreover, P has a diagonal matrix for the standard basis of $\mathbb{C}^4$: $P = diag\{c_1, c_2, c_3, c_4\}$, with $c_j = \sum_{\alpha,\beta,\gamma} E_{j\alpha\beta\gamma} E_{\alpha\beta\gamma j}$, and therefore $c_1 = c_4 = -(5q^3 + q^5)$, $c_2 = c_3 = -(2q^3 + 4q^5)$. Hence, for $q \neq 0, -1, 1$, which shall be the case in the sequel, $c_1 \neq c_2$, so P is not a multiple of the identity operator I. The condition $(P \otimes I)\mathbf{u} = \mathbf{u}(P \otimes I)$ is equivalent to $c_j \cdot u_{jk} = c_k \cdot u_{jk}$ for all natural numbers $1 \le j,k \le 4$. This yields $u_{12} = u_{21} = 0, u_{13} = u_{31} = 0, u_{24} = u_{42} = 0, u_{34} = u_{43} = 0$, and therefore

$$\mathbf{u} = \begin{pmatrix} u_{11} & 0 & 0 & u_{14} \\ 0 & u_{22} & u_{23} & 0 \\ 0 & u_{32} & u_{33} & 0 \\ u_{41} & 0 & 0 & u_{44} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 & b \\ 0 & x & y & 0 \\ 0 & z & w & 0 \\ c & 0 & 0 & d \end{pmatrix}. \tag{2.4}$$

This yields the decomposition of u decomposes into two irreducible subrepresentations

$$\mathbf{u} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigoplus \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \tag{2.5}$$

Substitution in (2.3) of appropriate sequences $(\alpha, \beta, \gamma, \delta)$ gives the following relations between the generators of the C^*-algebra $\mathcal{A}$ (the associated sequence is left of the relation):

$$\begin{array}{llllll}
(1423) & I = (ad - qbc)(xw - q^{-1}yz) & (1) & (4123) & I = (da - q^{-1}cb)(xw - q^{-1}yz) & (2) \\
(1432) & I = (ad - qbc)(wx - qzy) & (3) & (4132) & I = (da - q^{-1}cb)(wx - qzy) & (4) \\
(2314) & I = (xw - q^{-1}yz)(ad - qbc) & (5) & (2341) & I = (xw - q^{-1}yz)(da - q^{-1}cb) & (6) \\
(3214) & I = (wx - qzy)(ad - qbc) & (7) & (3241) & I = (wx - qzy)(da - q^{-1}cb) & (8)
\end{array}$$

Let $W = ad - qbc$ and $V = xw - q^{-1}yz$, then the above relation give $VW = I = WV$ and also $W = da - q^{-1}cb$, $V = wx - qzy$. Hence these relations are pairwise equivalent: (1) $\Leftrightarrow$ (5), (2) $\Leftrightarrow$ (6), (3) $\Leftrightarrow$ (7) and (4) $\Leftrightarrow$ (8). The operators V, W, being the inverse of each other, are twisted determinants for the two matrix co-representations:

$$W = \det_q \begin{pmatrix} a & b \\ c & d \end{pmatrix}, V = \det_{q^{-1}} \begin{pmatrix} x & y \\ z & w \end{pmatrix}. \tag{2.6}$$

Let us observe here that a change of order in the basis for $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$ gives us the matrix $\begin{pmatrix} w & z \\ y & x \end{pmatrix}$ which satisfies the same relations and for which the twisted determinant is

$$\det_q \begin{pmatrix} w & z \\ y & x \end{pmatrix} = wx - qzy = V. \tag{2.7}$$

Using the invertibility of W and V one can easily get the following relations:

$$\begin{array}{llllll}
(1123) & ab = qba & (9) & (2214) & cd = qdc & (10) \\
(4423) & yx = qxy & (11) & (3314) & wz = qzw & (12)
\end{array}$$

In addition, the relations (2.2) can be written as:

$$\begin{array}{llll}
I = aa^* + bb^* & (13) & I = cc^* + dd^* & (14) \\
I = a^*a + c^*c & (15) & I = b^*b + d^*d & (16) \\
0 = a^*b + c^*d & (17) & 0 = ca^* + db^* & (18)
\end{array}$$

and

$$\begin{array}{llll}
I = xx^* + yy^* & (19) & I = zz^* + ww^* & (20) \\
I = x^*x + z^*z & (21) & I = y^*y + w^*w & (22) \\
0 = x^*y + z^*w & (23) & 0 = zx^* + wy^* & (24)
\end{array}$$

Multiplication of (16) from the left by a^* and using (9) and then (17) gives the equation $d^*W = a$, or, equivalently, $d = V^*a^*$. On the other hand, multiplication (15) from the right by d and using (10) and (17) gives $d = a^*W$. These two combined ensure also that $W^*a = aV$. Similarly, by multiplying (16) from the right by c and using (10) and then (17) one gets $b^*W = -qc$, or equivalently, $b^* = -qcV$. Then, multiplying (15) from the right by b and using (9) and (17) one obtains $b = -qc^*W$. These two yield also $cV = W^*c$. Therefore we have

$$d = V^*a^* = a^*W, \quad b = -qV^*c^* = -qc^*W; \tag{2.8}$$

$$x = w^*V = W^*w^*, \quad z = -qy^*V = -qW^*y^*. \tag{2.9}$$

There are also other relations obtained from (2.3). They are listed in the following, with the associated sequences $(\alpha\beta\gamma\delta)$ on the left-hand side:

$$\begin{array}{lll}
(2143) & I = x(ad - qbc)w - qy(ad - q^{-1}bc)z & (25) \\
(2413) & I = x(da - q^{-1}cb)w - q^{-1}y(da - qcb)z & (26) \\
(3142) & I = w(ad - q^{-1}bc)x - q^{-1}z(ad - qbc)y & (27) \\
(3412) & I = w(da - qcb)x - qz(da - q^{-1}cb)y & (28)
\end{array}$$

and

$$\begin{array}{lll}
(1234) & I = a(xw - qyz)d - qb(xw - qyz)c & (29) \\
(4231) & I = d(xw - qyz)a - q^{-1}c(xw - qyz)b & (30) \\
(1324) & I = a(wx - q^{-1}zy)d - qb(wx - q^{-1}zy)c & (31) \\
(4321) & I = d(wx - q^{-1}zy)a - q^{-1}c(wx - q^{-1}zy)b & (32)
\end{array}$$

From now on we shall assume the following additional relation:

$$V = W^* \tag{2.10}$$

meaning that the twisted determinants are unitary operators. This yields that we are dealing with the quantum groups $U_q(2)$ (for the generators a, b, c, d) and another copy of $U_q(2)$ (for the generators w, y, z, x). This assumption is also necessary to allow the technical procedure used in [W3].

Let us substitute (2.8) into the (1) - (32). In (1) - (8) we do the substitution in one of the bracket and put V or V^* for the other. Thus for each equation we get two:

$$\begin{array}{rlrl}
VaV^*a^* + q^2c^*c = 1 & (1'a) & w^*VwV^* + yy^* = 1 & (1'b) \\
a^*a + VcV^*c^* = 1 & (2'a) & w^*VwV^* + yy^* = 1 & (2'b) \\
VaV^*a^* + q^2c^*c = 1 & (3'a) & ww^* + q^2y^*VyV^* = 1 & (3'b) \\
a^*a + VcV^*c^* = 1 & (4'a) & ww^* + q^2y^*VyV^* = 1 & (4'b)
\end{array} \tag{2.11}$$

We see that $(1'a) \Leftrightarrow (3'a)$, $(2'a) \Leftrightarrow (4'a)$, $(1'b) \Leftrightarrow (2'b)$ and $(3'b) \Leftrightarrow (4'b)$. For (9) – (12) we obtain:

$$\begin{array}{rlrl}
cVa^* = qa^*cV & (9') & aVc^* = qc^*aV & (10') \\
yw^*V = qw^*Vy & (11') & wy^*V = qy^*Vw & (12')
\end{array} \tag{2.12}$$

The relation (13) – (18) give:

$$\begin{array}{rlrl}
aa^* + q^2V^*c^*cV = 1 & (13') & cc^* + V^*a^*aV = 1 & (14') \\
a^*a + c^*c = 1 & (15') & aa^* + q^2cc^* = 1 & (16') \\
aVc = qcVa & (17') & Vca^* = qa^*cV & (18')
\end{array} \tag{2.13}$$

and for (19) – (24) we get:

$$\begin{array}{rlrl}
w^*w + yy^* = 1 & (19') & ww^* + q^2y^*y = 1 & (20') \\
ww^* + q^2yy^* = 1 & (21') & w^*w + y^*y = 1 & (22') \\
wy = qyw & (23') & wy^* = qy^*w & (24')
\end{array} \tag{2.14}$$

Let us first deal with the relations (2.14) involving w and y. Comparing (19′) with (21′) one gets easily that y is normal: $yy^* = y^*y$. Comparing (3′b) with (20′) gives

$$y^*Vy = y^*yV \tag{2.15}$$

and (1′b) with (19′) yield

$$w^*Vw = w^*wV. \tag{2.16}$$

Putting (24′) into (11′) gives

$$w^*yV = w^*Vy. \tag{2.17}$$

Multiplying both sides of (2.16) this from the left by w provides $ww^*yV = ww^*Vy$. Similarly, multiplying (2.14) from the right by y gives $yy^*Vy = yy^*yV$. Adding these two side by side yields

$$Vy = yV. \tag{2.18}$$

In a similar manner one gets

$$Vw = wV. \tag{2.19}$$

This requires putting (24′) into (12′) to get $y^*wV = y^*Vw$ which is then multiplied from the left by q^2y and added side by side to $ww^*Vw = ww^*wV$, which is obtained from (2.15). These can be collected together as the following relations:

$$\begin{array}{rr} w^*w + y^*y = 1 & ww^* + q^2yy^* = 1 \\ wy = qyw & wy^* = qy^*w \\ yy^* = y^*y & \\ wV = Vw & yV = Vy \end{array} \tag{2.20}$$

The fundamental co-representation is thus $\begin{pmatrix} w^*V & y \\ -qy^*V & w \end{pmatrix}$ and the above relations define the C^*-algebra of $U_q(2)$, and V is the $(-q)^{-1}$-determinant.

Let us now work with the relations for a and c. From (4') and (15') one deduces that $cVc^* = c^*cV$. Then, multiplying (9') from the right by a one gets $cVaa^* = qa^*cVa$. The left-hand side of this can be transformed as follows (using (15')):

$$cVaa^* = cV(1 - c^*c) = cV - (cVc^*)c = cV - c^*cVc.$$

For the right-hand side one can use (17') and then (15') to get:

$$qa^*cVa = a^*aVc = (1 - cc^*)Vc = Vc - c^*cVc.$$

It follows from these two that $cV = Vc$, and also $c^*V = Vc^*$, since V is unitary. Using this combined with (14') and (15') one obtains $cc^* = c^*c$, so c is normal. Then from (10') follows $ac^* = qc^*a$. Comparing (1'a) with (16') one concludes $aVa^* = aa^*V$. Then, multiplication of (17') by c^* from the right gives $aVcc^* = qcVac^*$. The left-hand side of this is $aV - aa^*Va$. The right-hand side of this can be transformed, with the help of the above relations, into:

$$qcVac^* = q^2cVc^*a = q^2c^*cVa = Va - aa^*Va.$$

Hence one concludes $aV = Va$, and also $a^*V = Va^*$. Therefore the above relations may be written as follows:

$$\begin{array}{ll} a^*a + c^*c = 1 & aa^* + q^2cc^* = 1 \\ ac = qca & ac^* = qc^*a \\ cc^* = c^*c & \\ aV = Va & cV = Vc \end{array} \tag{2.21}$$

For $N = 4$ we have more nontrivial relations between a, c, w, y given by (2.3) then in the case $N = 3$, since, for example the sequence $(1,1,2,2)$ gives a nontrivial relation here, and gave trivial relation there. Let us write them as follows, indicating the associated sequence $(\alpha, \beta, \gamma, \delta)$ on the left-hand side of it and successive numbering on the right-hand side of it. In the first set of equations we put elements from the same C^*-subalgebra outside, and the other inside.

$$\begin{array}{lcr} (1231) & a(xw - qyz)b = qb(xw - qyz)a & (33) \\ (1321) & a(wx - \frac{1}{q}zy)b = qb(wx - \frac{1}{q}zy)a & (34) \\ (4234) & c(xw - qyz)d = qd(xw - qyz)c & (35) \\ (4324) & c(wx - \frac{1}{q}zy)d = qd(wx - \frac{1}{q}zy)c & (36) \\ (2142) & x(ad - qbc)y = qy(ad - \frac{1}{q}bc)x & (37) \\ (2412) & y(da - qcb)x = qx(da - \frac{1}{q}cb)y & (38) \\ (3143) & z(ad - qbc)w = qw(ad - \frac{1}{q}bc)z & (39) \\ (3413) & w(da - qcb)z = qz(da - \frac{1}{q}cb)w & (40) \\ (1224) & axyd = qbxyc, \ ayxd = qbyxc & (41) \\ (4221) & cxyb = qdxya, \ cyxb = qdyxa & (42) \\ (1334) & azwd = qbzwc, \ awzd = qbwzc & (43) \\ (4331) & czwb = qdzwa, \ cwzb = qdwza & (44) \\ (2113) & yabz = 0 = ybaz & (45) \\ (3112) & wabx = 0 = wbax & (46) \\ (2443) & ydcz = 0 = ycdz & (47) \\ (3442) & wdcx = 0 = wcdx & (48) \end{array} \tag{2.22}$$

In the second set of equations we have alternating sequences of elements from different C^*-subalgebras.

$$\begin{array}{lcr} (1243) & axdw - qaydz - qbxcw + q^2bycz = I & (49) \\ (4213) & dxaw - qdyaz - \frac{1}{q}cxbw + cybz = I & (50) \\ (1342) & awdx - \frac{1}{q}azdy - qbwcx + bzcy = I & (51) \\ (4312) & dwax - \frac{1}{q}dzay - \frac{1}{q}cwbx + \frac{1}{q^2}czby = I & (52) \\ (2134) & xawd - qxbwc - qyazd + q^2ybzc = I & (53) \\ (3124) & waxd - qwbxc - \frac{1}{q}zayd + zbyc = I & (54) \\ (2431) & xdwa - \frac{1}{q}xcwb - qydza + yczb = I & (55) \\ (3421) & wdxa - \frac{1}{q}wcxb - \frac{1}{q}zdya + \frac{1}{q^2}zcyb = I & (56) \end{array} \tag{2.23}$$

Computing

$$\begin{array}{rcl} xw - qyz & = & V - (1 - q^2)yy^*V \\ wx - \frac{1}{q}zy & = & V + (1 - q^2)yy^*V \\ ad - \frac{1}{q}bc & = & V^* + (1 - q^2)cc^*V^* \\ da - qcb & = & V^* - (1 - q^2)cc^*V^* \end{array} \tag{2.24}$$

and substituting these into (2.22) one obtains

$$\begin{array}{rcll} ayy^*c^* &=& qc^*yy^*a & (33'), (34') \\ cyy^*a^* &=& qa^*yy^*c & (35'), (36') \\ w^*y &=& qycc^*w^* & (37'), (39') \\ ycc^*w^* &=& 0 & (38') \\ wcc^*y^* &=& 0 & (40') \\ aw^*ya^* + q^2c^*w^*yc &=& 0 & (41'), (43') \\ a^*w^*ya + cw^*yc^* &=& 0 & (42'), (44') \\ yac^*y^* &=& 0 & (45') \\ wac^*w^* &=& 0 & (46') \\ ya^*cy^* &=& 0 & (47') \\ wa^*cw^* &=& 0 & (48') \end{array} \qquad (2.25)$$

Unfortunately, (37') combined with (38') give

$$w^*y = 0$$

and it follows from (2.20) that $y = 0$. To see this let us observe that $ww^*yy^* + q^2yy^*yy^* = yy^*$ implies $q^2(yy^*)^2 = yy^*$, and hence, by induction, $q^{2n}(yy^*)^{n+1} = yy^*$ for any positive integer $n \in \mathbb{N}$. This yields that the spectral radius $r(yy^*) = \lim_n \|(yy^*)^n\|^{\frac{1}{n}}$ satisfies $r(yy^*) = q^{-2} > 1$. However, it follows from the description of the irreducible representations of the relations (2.20) (see [W3]) that $\|y\| \leq 1$, so that $r(yy^*) \leq 1$. This is a contradiction, except $y = 0$.

Then $xw = V = wx$ and $xx^* = 1 = x^*x$, $ww^* = 1 = w^*w$, so that x, w are unitary. Moreover $x = w^*V$, so that for the fundamental co-representation eventually we get $\begin{pmatrix} w^*V & 0 \\ 0 & w \end{pmatrix}$. In a similar manner one gets that

$$a^*c = 0$$

and hence $c = 0$. Substitution of these to (2.23) gives

$$awa^*w^* = 1 = a^*w^*aw.$$

If we set $t := aw$ and $s := wa$, then $tt^* = 1 = t^*t$, $ss^* = 1 = s^*s$ and $ts^* = 1 = s^*t$. Therefore $t = s$, which gives $aw = wa$.

These computations show that the C^*-algebra of the constructed quantum group is generated by three commuting unitaries a, w, V, so it is isomorphic to $C(\mathbb{T}) \otimes C(\mathbb{T}) \otimes C(\mathbb{T})$. Therefore, the quantum group we consider is in fact the classical group $U(1) \times U(1) \times U(1)$.

3 The Yang-Baxter operator associated with $U_q(2)$

In the next two Sections we are going to show a construction of a cubic Hecke algebra associated with the quantum group $U_q(2)$. In [W3] we gave a construction of the quantum group $U_q(2)$, in which the crucial role is played by the function counting the number of cycles in permutations from the symmetric group S_3. Namely, by considering the function $S_3 \ni \sigma \mapsto (-q)^{3-c(\sigma)}$, where $c(\sigma)$ is the number of cycles and $q > 0$, we constructed the following array:

$$\begin{array}{ll} E_{1,2,3} = 1 & E_{1,3,2} = E_{2,1,3} = E_{3,2,1} = -q \\ E_{2,3,1} = E_{3,1,2} = q^2 & E_{i,j,k} = 0 \text{ if } \{i,j,k,\} \subsetneq \{1,2,3\} \end{array}$$

This array defines an operator ρ on $\mathbb{C}^3 \otimes \mathbb{C}^3$ by

$$\rho: \ \mathbb{C}^3 \otimes \mathbb{C}^3 \ni (a,b) \mapsto \sum_{i,j,k=1}^{3} E_{i,j,k} E_{k,a,b}(i,j) \in \mathbb{C}^3 \otimes \mathbb{C}^3, \tag{3.26}$$

where (a,b) denotes in short the standard basis element $\varepsilon_a \otimes \varepsilon_b$. In particular $\varepsilon_1 = (1,0,0)$, $\varepsilon_2 = (0,1,0)$ and $\varepsilon_3 = (0,0,1)$.

The definition of E implies that (3.26) simplifies to

$$\rho(a,b) = E_{a,b,k} E_{k,a,b}(a,b) + E_{b,a,k} E_{k,a,b}(b,a), \quad \text{where} \quad \{a,b,k\} = \{1,2,3\} \tag{3.27}$$

for $a \neq b$ and $a,b = 1,2,3$. If $a = b$ then we get $\rho(a,a) = 0$. The formulas can be written explicitely as follows.

$$\begin{array}{rcccl}
\rho(1,2) & = & E_{1,2,3}E_{3,1,2}(1,2) + E_{2,1,3}E_{3,1,2}(2,1) & = & q^2(1,2) + q^3(2,1) \\
\rho(2,1) & = & E_{2,1,3}E_{3,2,1}(2,1) + E_{1,2,3}E_{3,2,1}(1,2) & = & q^2(2,1) + q(1,2) \\
\rho(1,3) & = & E_{1,3,2}E_{2,1,3}(1,3) + E_{3,1,2}E_{2,1,3}(3,1) & = & q^2(1,3) + q^3(3,1) \\
\rho(3,1) & = & E_{3,1,2}E_{2,3,1}(3,1) + E_{1,3,2}E_{2,3,1}(1,3) & = & q^4(3,1) + q^3(1,3) \\
\rho(2,3) & = & E_{2,3,1}E_{1,2,3}(2,3) + E_{3,2,1}E_{1,2,3}(3,2) & = & q^2(2,3) + q(3,2) \\
\rho(3,2) & = & E_{3,2,1}E_{1,3,2}(3,2) + E_{2,3,1}E_{1,3,2}(2,3) & = & q^2(3,2) + q^3(2,3)
\end{array}$$

Therefore, the operator $\alpha := I_2 - \dfrac{1}{q^2}\rho$ acts as: $\alpha(a,a) = (a,a)$ for $a = 1,2,3$ and

$$\begin{array}{rcl}
\alpha(1,2) & = & -q(2,1) \\
\alpha(1,3) & = & -q(3,1) \\
\alpha(3,2) & = & -q(2,3) \\
\alpha(2,1) & = & -q^{-1}(1,2) \\
\alpha(2,3) & = & -q^{-1}(3,2) \\
\alpha(3,1) & = & (1-q^2)(3,1) - q(1,3)
\end{array} \tag{3.28}$$

This operator is not self-adjoint, but $\alpha^2 = (\alpha^2)^*$ is so, since

$$\begin{array}{rcl}
\alpha^2(1,2) & = & (2,1) \\
\alpha^2(2,1) & = & (2,1) \\
\alpha^2(2,3) & = & (3,2) \\
\alpha^2(3,2) & = & (2,3) \\
\alpha^2(1,3) & = & q^2(1,3) - q(1-q^2)(3,1) \\
\alpha^2(3,1) & = & (1-q^2+q^4)(3,1) - q(1-q^2)(1,3)
\end{array} \tag{3.29}$$

The first important property of α is that it is a Yang-Baxter operator.

Proposition 3.1 *The operator α satisfies the Yang-Baxter equation*

$$(\alpha \otimes I)(I \otimes \alpha)(\alpha \otimes I) = (I \otimes \alpha)(\alpha \otimes I)(I \otimes \alpha). \tag{3.30}$$

Proof: Let $L = (\alpha\otimes I)(I\otimes\alpha)(\alpha\otimes I)$ be the left-hand side and $P = (I\otimes\alpha)(\alpha\otimes I)(I\otimes\alpha)$ be the right-hand side of (3.30). We have to show that $L(a,b,c) = P(a,b,c)$ for every $a,b,c \in \{1,2,3\}$ (with the notation: $(a,b,c) = \varepsilon_a \otimes \varepsilon_b \otimes \varepsilon_c$). This requires checking 27 cases. It is clear thet $L(a,a,a) = (a,a,a) = P(a,a,a)$ for any $a = 1,2,3$. The direct calculation provides the following formulas for the other cases.

$$\begin{array}{rclcl}
L(3,2,3) & = & (3,2,3) & = & P(3,2,3)\\
L(2,3,2) & = & (2,3,2) & = & P(2,3,2)\\
L(1,2,1) & = & (1,2,1) & = & P(1,2,1)\\
L(2,1,2) & = & (2,1,2) & = & P(2,1,2)\\
L(1,2,3) & = & -q(3,2,1) & = & P(1,2,3)\\
L(1,3,2) & = & -q^3(2,3,1) & = & P(1,3,2)\\
L(2,1,3) & = & -q^{-1}(3,1,2) & = & P(3,1,2)\\
L(3,3,2) & = & q^2(2,3,3) & = & P(3,3,2)\\
L(2,2,3) & = & q^2(3,2,2) & = & P(2,2,3)\\
L(3,2,2) & = & q^2(2,2,3) & = & P(3,2,2)\\
L(1,1,3) & = & q^2(3,1,1) & = & P(1,1,3)\\
L(1,3,3) & = & q^2(3,3,1) & = & P(1,3,3)\\
L(1,1,2) & = & q^2(2,1,1) & = & P(1,1,2)\\
L(1,2,2) & = & q^2(2,2,1) & = & P(1,2,2)\\
L(2,3,3) & = & q^{-2}(3,3,2) & = & P(2,3,3)\\
L(2,1,1) & = & q^{-2}(1,1,2) & = & P(2,1,1)\\
L(2,2,1) & = & q^{-2}(1,2,2) & = & P(2,2,1)
\end{array} \tag{3.31}$$

$$\begin{array}{rclclcl}
L(3,2,1) & = & (1-q^2)(3,2,1) & - & q(1,2,3) & = & P(3,2,1)\\
L(3,1,2) & = & q^2(1-q^2)(2,3,1) & - & q^3(2,1,3) & = & P(3,1,2)\\
L(2,3,1) & = & q^{-2}(1-q^2)(3,1,2) & - & q^{-1}(1,3,2) & = & P(2,3,1)\\
L(1,3,1) & = & -q(1-q^2)(3,1,1) & + & q^2(1,3,1) & = & P(1,3,1)\\
L(3,1,3) & = & -q(1-q^2)(3,3,1) & + & q^2(3,1,3) & = & P(3,1,3)
\end{array} \tag{3.32}$$

$$\begin{array}{rclclclcl}
L(3,1,1) & = & (1-q^2)(3,1,1) & - & q(1-q^2)(1,3,1) & + & q^2(1,1,3) & = & P(3,1,1)\\
L(3,3,1) & = & (1-q^2)(3,3,1) & - & q(1-q^2)(3,1,3) & + & q^2(1,3,3) & = & P(3,3,1)
\end{array} \tag{3.33}$$

From these formulas the Proposition follows. □

4 The cubic Hecke algebra associated with $U_q(2)$

The second important property of the operator α is that, even though it is not a Hecke operator, it does satisfy a cubic equation, and thus it generates a *cubic Hecke algebra.* This notion has been introduced by Funar in [F], where the cubic equation $\alpha^3 - I = 0$ was considered.

Proposition 4.1 *The operator α satisfies the cubic equation:*

$$(\alpha^2 - I)(\alpha + q^2 I) = 0. \tag{4.34}$$

Proof: From the formulas (3.28), defining α it follows that it acts on the following subspaces by simple matricial formulas.

1. On the span of $(1,2),(2,1)$ as $\beta := \begin{pmatrix} 0 & \frac{-1}{q} \\ -q & 0 \end{pmatrix}$
2. On the span of $(2,3),(3,2)$ as $\beta^* := \begin{pmatrix} 0 & -q \\ \frac{-1}{q} & 0 \end{pmatrix}$
3. On the span of $(1,3),(3,1)$ as $\gamma := \begin{pmatrix} 0 & -q \\ -q & 1-q^2 \end{pmatrix}$
4. As identity on every (a,a) with $a = 1,2,3$.

It is strightforward to see that $\beta^2 - I = 0 = (\beta^*)^2 - I$. On the other hand, since

$$\gamma^2 = \begin{pmatrix} q^2 & -q(1-q^2) \\ -q(1-q^2) & 1-q^2+q^4 \end{pmatrix},$$

we obtain

$$(\gamma^2 - I)(\gamma + q^2 I) = (q^2-1)\begin{pmatrix} 1 & q \\ q & q^2 \end{pmatrix}\begin{pmatrix} q^2 & -q \\ -q & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore both β and γ satisfy the equation (4.34), so the α does. □

Let us define the elements

$$h_j := I_j \otimes \alpha \otimes I_{n-j-2} \quad \text{for} \quad j = 1, \ldots, n-2, \tag{4.35}$$

where I_k denotes the identity map on $(\mathbb{C}^N)^{\otimes k}$. Then by Propositions 3.1 and 4.1 the elements $h_1, \ldots, h_n$ generate a cubic Hecke algebra, associated with the quantum group $U_q(2)$.

Definition 4.2 *The algebra $\mathcal{H}_{q,n}(2)$ generated by the elements h_j, $j = 1,2,\ldots,n$ defined by* (4.35) *will be called the* ***cubic Hecke algebra*** *associated with the quantum group $U_q(2)$.*

The basic properties of this algebra are summarized in the following.

Theorem 4.3 *The generators $\{h_j \ : \ 1 \le j \le n\}$ of $H_{q,n}(2)$ satisfy:*

$$\begin{aligned} h_j h_{j+1} h_j &= h_{j+1} h_j h_{j+1} && \text{for } j = 1,\ldots,n-1, \\ h_j h_k &= h_k h_j && \text{for } |j-k| \ge 2, \\ ((h_j)^2 - 1)(h_j + q^2) &= 0 && \text{for } j = 1,\ldots,n. \end{aligned} \tag{4.36}$$

The role of the Hecke algebra in the study of $SU_q(N)$ was that it was the intertwining algebra of the tensor powers of the fundamental co-representation. In [W3] the irreducible co-representations of $U_q(2)$ have been described, but it is not clear if the description is complete. So, it is still to be checked whether $\mathcal{H}_{q,n}(2)$ plays the same role as in $SU_q(N)$.

References

[F] L. FUNAR *On the quotients of cubic Hecke algebras*, Comm.Math.Phys. **173** (1995), 513–558.

[K] H. T. KOELINK, *On $*$-representatons of the Hopf $*$-algebra associated with the quantum group $U_q(n)$*, Compositio Math. **77** (1991), 199–231.

[M-HR] F. MÜLLER-HOISSEN, C. REUTEN *Bicovariant differential calculi on $GL_{p,q}(2)$ and quantum subgroups*, J. Phys. A **26** (1993), 2955 - 2975.

[PoM] P. PODLEŚ, E. MÜLLER, *Introduction to quantum groups*, Rev. Math. Phys. **10** (1998), no. 4, 511–551.

[PoW] P. PODLEŚ, S.L. WORONOWICZ *Quantum deformation of Lorentz group*, Comm. Math. Phys. **130** (1990), 381–431.

[PuW] W. PUSZ, S. L. WORONOWICZ, *Representations of quantum Lorentz group on Gelfand spaces*, Rev. Math. Phys. Vol. **12**, No. 12 (2000), 1551 – 1625.

[Ta] M. TAKEUCHI, *A two-parameter quantization of* $GL(n)$, Proc. Japan Acad. Ser. A Math. Sci **66** no. 5 (1990), 112 – 114.

[SLW1] S.L. WORONOWICZ, *Twisted SU(2) group. An example of non-commutative differential calculus*, Publ. RIMS, Kyoto Univ. **23** (1987), 117–181.

[SLW2] S.L. WORONOWICZ, *Compact Matrix Pseudogroups*, Comm. Math. Phys. **111** (1987), 613–665

[SLW3] S.L. WORONOWICZ, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted SU(N) groups*, Invent. Math. **93** (1988), 35–76.

[SLW4] S.L. WORONOWICZ, *Compact quantum groups*, Symétries quantiques (Les Houches, Session LXIV), (1995), 845 – 884; in "Quantum symmetries" (A. Connes, K. Gawędzki, J. Zinn-Justin, Eds.) (1998) North-Holland, Amsterdam.

[SLW5] S.L. WORONOWICZ, *From multiplicative unitaries to quantum groups*, Internat. J. Math. **7** (1996), 127 - 149.

[W1] J. WYSOCZAŃSKI, *A construction of compact matrix quantum groups and description of the related C^*-algebras*, Infinite dimensional analysis and quantum probability theory (Japanese) (Kyoto, 2000). Su-rikaisekikenkyu-sho Ko-kyu-roku No. 1227 (2001), 209–217.

[W2] J. WYSOCZAŃSKI, *Unitary representations for twisted product of matrix quantum groups*, Trends in infinite-dimensional analysis and quantum probability (Japanese) (Kyoto, 2001). Su-rikaisekikenkyu-sho Ko-kyu-roku No. 1278 (2002), 188–193.

[W3] J. WYSOCZAŃSKI, *Twisted product structure and representation theory of the quantum group $U_q(2)$*, Reports on Math. Phys. **54** No.3 (2004), 327–347.